

Quasideterminant solutions of a non-Abelian Toda lattice and kink solutions of a matrix sine-Gordon equation

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Abstract

Two families of solutions of a generalized non-Abelian Toda lattice are considered. These solutions are expressed in terms of quasideterminants, constructed by means of Darboux and binary Darboux transformations. As an example of the application of these solutions, we consider the 2-periodic reduction to a matrix sine-Gordon equation. In particular, we investigate the interaction properties of polarized kink solutions.

1 Introduction

There has been great interest in noncommutative versions of some well-known soliton equations, such as the KP equation, the KdV equation and the Hirota-Miwa equation [1–12]. Often, these noncommutative versions are obtained simply by removing the assumption that the coefficients in the Lax pair of the commutative equation commute.

The non-Abelian Toda lattice

$$U_{n,x} + U_n V_{n+1} - V_n U_n = 0, \quad (1.1)$$

$$V_{n,t} + U_{n-1} - U_n = 0, \quad (1.2)$$

was first studied in [13]. A Darboux transformations for this system was given by [14]. In [15], the following generalization

$$U_{n,x} + U_n V_{n+1} - V_n U_n = 0, \quad (1.3)$$

$$V_{n,t} + \alpha_n U_{n-1} - U_n \alpha_{n+1} = 0, \quad (1.4)$$

was studied and the Darboux and binary Darboux transformations were obtained. We note that in general, α_n is not a scalar, but is independent of t . In the case that U_n , V_n and α_n are scalars, it is easy to show, by setting $\alpha_{n+1} U_n = e^{-\theta_n}$ and eliminating V_n , that (1.3)–(1.4) becomes the standard two-dimensional Toda lattice equation

$$\theta_{n,x} - e^{-\theta_{n-1}} + 2e^{-\theta_n} - e^{-\theta_{n+1}} = 0. \quad (1.5)$$

Introducing new variables X_n where

$$U_n = X_n X_{n+1}^{-1}, \quad V_n = X_{n,x} X_n^{-1}, \quad (1.6)$$

(1.3)–(1.4) can be rewritten as

$$(X_{n,x} X_n^{-1})_t + \alpha_n X_{n-1} X_n^{-1} - X_n X_{n+1}^{-1} \alpha_{n+1} = 0. \quad (1.7)$$

From now on, we will refer to (1.7) as the non-Abelian Toda lattice. One type of quasideterminant solutions of (1.7) were found in [16]. We will show how these (quasiwronskian) solutions arise from the Darboux transformation and consider a second type of quasideterminant, which we call quasigrammian, solutions obtained using the binary Darboux transformation.

It is well known that the 2-periodic reduction of the standard two dimensional Toda lattice leads to the scalar sine-Gordon equation. In the same way, the 2-periodic reduction of the non-Abelian Toda lattice (1.7) leads to a noncommutative sine-Gordon equation. This equation has been studied already in a number of papers [16–23] concerning both the matrix and the Moyal product versions. Here we only consider in detail the matrix version.

Recently, a matrix KdV equation was considered in [24]. A multisoliton solution was found by using the inverse scattering method. In particular, the properties of one- and two-soliton solutions expressed in terms of projection matrices were investigated. We will apply some of these ideas to the matrix sine-Gordon equation to study the interaction of its kink solutions.

The paper is organized as follows. In Section 2, some properties of quasideterminants used in the paper are described. In Section 3, we present quasiwronskian solutions to the non-Abelian Toda lattice constructed by iterating Darboux transformations and in Section 4, we present quasigrammian solutions to the system by using the related binary Darboux transformation. In the rest of the paper we consider the 2-periodic reduction to a matrix sine-Gordon equation. In particular we consider the matrix kink solutions obtained from the quasigrammian solutions, we show that kink solutions for the matrix sine-Gordon equation emerge intact from interaction apart from change of polarization and phase.

2 Preliminaries

In this short section we recall some of the key elementary properties of quasideterminants. The reader is referred to the original papers [16, 25, 26] for a more detailed and general treatment.

2.1 Quasideterminants

An $n \times n$ matrix A over a ring \mathcal{R} (noncommutative, in general) has n^2 *quasideterminants* written as $|A|_{i,j}$ for $i, j = 1, \dots, n$, which are also elements of \mathcal{R} . They are defined recursively by

$$|A|_{i,j} = a_{i,j} - r_i^j (A^{i,j})^{-1} c_j^i, \quad A^{-1} = (|A|_{j,i}^{-1})_{i,j=1,\dots,n}. \quad (2.1)$$

In the above r_i^j represents the i th row of A with the j th element removed, c_j^i the j th column with the i th element removed and $A^{i,j}$ the submatrix obtained by removing the i th row and the j th column from A . Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made

$$|A|_{i,j} = \begin{vmatrix} A^{i,j} & c_j^i \\ r_i^j & \boxed{a_{i,j}} \end{vmatrix}.$$

The case $n = 1$ is rather trivial; let $A = (a)$, say, and then there is one quasideterminant $|A|_{1,1} = \boxed{a} = a$. For $n = 2$, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then there are four quasideterminants

$$\begin{aligned} |A|_{1,1} &= \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c, & |A|_{1,2} &= \begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d, \\ |A|_{2,1} &= \begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a, & |A|_{2,2} &= \begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b. \end{aligned}$$

Note that if the entries in A commute, the above becomes the familiar formula for the inverse of a 2×2 matrix with entries expressed as ratios of determinants. Indeed this is true for any size of square matrix; if the entries in A commute then

$$|A|_{i,j} = (-1)^{i+j} \frac{\det(A)}{\det(A^{i,j})}. \quad (2.2)$$

In this paper we will consider only quasideterminants that are expanded about a term in the last column, most usually the last entry. For a block matrix

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix}$$

where $d \in \mathcal{R}$, A is a square matrix over \mathcal{R} of arbitrary size and B, C are column and row vectors over \mathcal{R} of compatible lengths, we have

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

2.2 Noncommutative Jacobi Identity

There is a quasideterminant version of Jacobi's identity for determinants, called the noncommutative Sylvester's Theorem by Gelfand and Retakh [25]. The simplest version of this identity is given by

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}. \quad (2.3)$$

As a direct result, we have the homological relation

$$\begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}. \quad (2.4)$$

2.3 Quasi-Plücker coordinates

Given an $(n+k) \times n$ matrix A , denote the i th row of A by A_i , the submatrix of A having rows with indices in a subset I of $\{1, 2, \dots, n+k\}$ by A_I and $A_{\{1, \dots, n+k\} \setminus \{i\}}$ by $A_{\bar{i}}$. Given $i, j \in \{1, 2, \dots, n+k\}$ and I such that $\#I = n-1$ and $j \notin I$, one defines the (*right*) *quasi-Plücker coordinates*

$$r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix} A_I \\ A_i \end{vmatrix}_{ns} \begin{vmatrix} A_I \\ A_j \end{vmatrix}_{ns}^{-1} = - \begin{vmatrix} A_I & 0 \\ A_i & \boxed{0} \\ A_j & 1 \end{vmatrix}, \quad (2.5)$$

for any column index $s \in \{1, \dots, n\}$. The final equality in (2.5) comes from an identity of the form (2.3) and proves that the definition is independent of the choice of s .

Remark. A useful consequence of (2.5) is the identity

$$\begin{vmatrix} A^I & 0 \\ A^i & \boxed{0} \\ A^j & 1 \end{vmatrix}^{-1} = \begin{vmatrix} A^I & 0 \\ A^i & 1 \\ A^j & \boxed{0} \end{vmatrix}, \quad (2.6)$$

which shows that quasideterminants of this form may be inverted very simply.

3 Solutions obtained by Darboux transformations

The non-Abelian Toda lattice (1.3)–(1.4) has Lax pair

$$\phi_{n,x} = V_n \phi_n + \alpha_n \phi_{n-1}, \quad (3.1)$$

$$\phi_{n,t} = U_n \phi_{n+1}. \quad (3.2)$$

Let $\theta_{n,i}, i = 1, \dots, N$ be a particular set of eigenfunctions of the linear system and introduce the notation $\Theta_n = (\theta_{n,1}, \dots, \theta_{n,N})$. The Darboux transformation, determined by particular solution θ_n , for the non-Abelian Toda lattice is

$$\tilde{\phi}_n = \phi_n - \theta_n \theta_{n+1}^{-1} \phi_{n+1}, \quad (3.3)$$

$$\tilde{V}_n = V_n + \alpha_n \theta_{n-1} \theta_n^{-1} - \theta_n \theta_{n+1}^{-1} \alpha_{n+1}, \quad (3.4)$$

$$\tilde{U}_n = U_n - (\theta_n \theta_{n+1}^{-1})_t = \theta_n \theta_{n+1}^{-1} U_{n+1} \theta_{n+2} \theta_{n+1}^{-1}, \quad (3.5)$$

$$\tilde{X}_n = \theta_n \theta_{n+1}^{-1} X_{n+1}. \quad (3.6)$$

This may be iterated by defining

$$\phi_n[k+1] = \phi_n[k] - \theta_n[k] \theta_{n+1}[k]^{-1} \phi_{n+1}[k], \quad (3.7)$$

$$X_n[k+1] = \theta_n[k] \theta_{n+1}[k]^{-1} X_{n+1}[k], \quad (3.8)$$

where $\phi_n[1] = \phi_n, X_n[1] = X_n$ and

$$\theta_n[k] = \phi_n[k] |_{\phi_n \rightarrow \theta_{n,k}}. \quad (3.9)$$

In particular,

$$\phi_n[2] = \phi_n - \theta_{n,1} \theta_{n+1,1}^{-1} \phi_{n+1}, \quad (3.10)$$

$$X_n[2] = \theta_{n,1} \theta_{n+1,1}^{-1} X_{n+1}. \quad (3.11)$$

In what follows, we will show by induction that the results of N repeated Darboux transformation $\phi_n[N+1]$ and $X_n[N+1]$ can be expressed as in closed form as quasideterminants

$$\phi_n[N+1] = \begin{vmatrix} \Theta_n & \boxed{\phi_n} \\ \Theta_{n+1} & \phi_{n+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \phi_{n+N} \end{vmatrix}, \quad X_n[N+1] = (-1)^N \begin{vmatrix} \Theta_n & \boxed{0} \\ \Theta_{n+1} & 0 \\ \vdots & \vdots \\ \Theta_{n+N} & 1 \end{vmatrix} X_{n+N}. \quad (3.12)$$

The initial case $N = 1$ follows directly from (3.10)–(3.11). Also using the noncommutative Jacobi identity (2.3) and the homological relation (2.4) we have

$$\begin{aligned}
\phi_n[N+2] &= \phi_n[N+1] - \theta_n[N+1]\theta_{n+1}[N+1]^{-1}\phi_{n+1}[N+1] \\
&= \begin{vmatrix} \Theta_n & \boxed{\phi_n} \\ \Theta_{n+1} & \phi_{n+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \phi_{n+N} \end{vmatrix} - \begin{vmatrix} \Theta_n & \boxed{\theta_{n,N+1}} \\ \Theta_{n+1} & \theta_{n+1,N+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \theta_{n+N,N+1} \end{vmatrix} \begin{vmatrix} \Theta_{n+1} & \boxed{\theta_{n+1,N+1}} \\ \Theta_{n+2} & \theta_{n+2,N+1} \\ \vdots & \vdots \\ \Theta_{n+N+1} & \theta_{n+N+1,N+1} \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1} & \boxed{\phi_{n+1}} \\ \Theta_{n+2} & \phi_{n+2} \\ \vdots & \vdots \\ \Theta_{n+N+1} & \phi_{n+N+1} \end{vmatrix} \\
&= \begin{vmatrix} \Theta_n & \boxed{\phi_n} \\ \Theta_{n+1} & \phi_{n+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \phi_{n+N} \end{vmatrix} - \begin{vmatrix} \Theta_n & \boxed{\theta_{n,N+1}} \\ \Theta_{n+1} & \theta_{n+1,N+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \theta_{n+N,N+1} \end{vmatrix} \begin{vmatrix} \Theta_{n+1} & \theta_{n+1,N+1} \\ \Theta_{n+2} & \theta_{n+2,N+1} \\ \vdots & \vdots \\ \Theta_{n+N+1} & \boxed{\theta_{n+N+1,N+1}} \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1} & \phi_{n+1} \\ \Theta_{n+2} & \phi_{n+2} \\ \vdots & \vdots \\ \Theta_{n+N+1} & \boxed{\phi_{n+N+1}} \end{vmatrix} \\
&= \begin{vmatrix} \Theta_n & \theta_{n,N+1} & \boxed{\phi_n} \\ \Theta_{n+1} & \theta_{n+1,N+1} & \phi_{n+1} \\ \vdots & \vdots & \vdots \\ \Theta_{n+N+1} & \theta_{n+N+1,N+1} & \phi_{n+N+1} \end{vmatrix}
\end{aligned}$$

and

$$\begin{aligned}
X_n[N+2] &= \theta_n[N+1]\theta_{n+1}[N+1]^{-1}X_{n+1}[N+1] \\
&= (-1)^N \begin{vmatrix} \Theta_n & \boxed{\theta_{n,N+1}} \\ \Theta_{n+1} & \theta_{n+1,N+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \theta_{n+N,N+1} \end{vmatrix} \begin{vmatrix} \Theta_{n+1} & \boxed{\theta_{n+1,N+1}} \\ \Theta_{n+2} & \theta_{n+2,N+1} \\ \vdots & \vdots \\ \Theta_{n+N+1} & \theta_{n+N+1,N+1} \end{vmatrix}^{-1} \begin{vmatrix} \Theta_{n+1} & \boxed{0} \\ \Theta_{n+2} & 0 \\ \vdots & \vdots \\ \Theta_{n+N+1} & 1 \end{vmatrix} X_{n+N+1} \\
&= (-1)^N \begin{vmatrix} \Theta_n & \boxed{\theta_{n,N+1}} \\ \Theta_{n+1} & \theta_{n+1,N+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \theta_{n+N,N+1} \end{vmatrix} \begin{vmatrix} \Theta_{n+1} & \theta_{n+1,N+1} \\ \vdots & \vdots \\ \Theta_{n+N} & \theta_{n+N,N+1} \\ \Theta_{n+N+1} & \boxed{\theta_{n+N+1,N+1}} \end{vmatrix}^{-1} X_{n+N+1}
\end{aligned}$$

and then using the quasi-Plücker coordinate formula (2.5), we get

$$= (-1)^{N+1} \begin{vmatrix} \Theta_n & \theta_{n,N+1} & \boxed{0} \\ \Theta_{n+1} & \theta_{n+1,N+1} & 0 \\ \vdots & \vdots & \vdots \\ \Theta_{n+N+1} & \theta_{n+N+1,N+1} & 1 \end{vmatrix} X_{n+N+1}.$$

This proves the inductive step and the proof is complete.

4 Solutions obtained by binary Darboux transformation

The linear equations (3.1) and (3.2) have the formal adjoints

$$-\psi_{n,x} = V_n^\dagger \psi_n + \alpha_{n+1}^\dagger \psi_{n+1}, \quad (4.1)$$

$$-\psi_{n,t} = U_{n-1}^\dagger \psi_{n-1}. \quad (4.2)$$

Following the standard construction of a binary Darboux transformation, one introduces a potential $\Omega_n = \Omega(\phi_n, \psi_n)$ satisfying the three conditions

$$\Omega(\phi_n, \psi_n)_x = \psi_{n+1}^\dagger \alpha_{n+1} \phi_n, \quad (4.3)$$

$$\Omega(\phi_n, \psi_n)_t = -\psi_n^\dagger U_n \phi_{n+1}, \quad (4.4)$$

$$\Omega_n - \Omega_{n-1} = -\psi_n^\dagger \phi_n. \quad (4.5)$$

A binary Darboux transformation is then defined by

$$\phi_n[N+1] = \phi_n[N] - \theta_n[N] \Omega(\theta_n[N], \rho_n[N])^{-1} \Omega(\phi_n[N], \rho_n[N]), \quad (4.6)$$

$$\psi_n[N+1] = \psi_n[N] - \rho_n[N] \Omega(\theta_{n-1}[N], \rho_{n-1}[N])^{-\dagger} \Omega(\theta_{n-1}[N], \psi_{n-1}[N])^\dagger, \quad (4.7)$$

$$X_n[N+1] = (I + \theta_n[N] \Omega(\theta_n[N], \rho_n[N])^{-1} \rho_n[N]^\dagger) X_n[N], \quad (4.8)$$

where $\phi_n[1] = \phi_n, \psi_n[1] = \psi_n, X_n[1] = X_n$ and

$$\theta_n[N] = \phi_n[N]|_{\phi_n \rightarrow \theta_{n,N}}, \quad \rho_n[N] = \psi_n[N]|_{\psi_n \rightarrow \rho_{n,N}} \quad (4.9)$$

Using the notation $\Theta_n = (\theta_{n,1}, \dots, \theta_{n,N})$ and $P_n = (\rho_{n,1}, \dots, \rho_{n,N})$, it is easy to prove by induction that for $N \geq 1$,

$$\phi_n[N+1] = \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\phi_n, P_n) \\ \Theta_n & \boxed{\phi_n} \end{vmatrix}, \quad (4.10)$$

$$\psi_n[N+1] = \begin{vmatrix} \Omega(\Theta_{n-1}, P_{n-1})^\dagger & \Omega(\Theta_{n-1}, \psi_{n-1})^\dagger \\ P_n & \boxed{\psi_n} \end{vmatrix} \quad (4.11)$$

and

$$\Omega(\phi_n[N+1], \psi_n[N+1]) = \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\phi_n, P_n) \\ \Omega(\Theta_n, \psi_n) & \boxed{\Omega(\phi_n, \psi_n)} \end{vmatrix}. \quad (4.12)$$

We may thus after N binary Darboux transformations we obtain

$$X_n[N+1] = - \begin{vmatrix} \Omega(\Theta_n, P_n) & P_n^\dagger \\ \Theta_n & \boxed{-I} \end{vmatrix} X_n. \quad (4.13)$$

In fact, we can prove the above results by induction.

$$\begin{aligned} X_n[N+2] &= (I + \Theta_n[N+1] \Omega(\Theta_n[N+1], P_n[N+1])^{-1} P_n[N+1]^\dagger) X_n[N+1] \\ &= - \left(I + \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\theta_{n,N+1}, P_n) \\ \Theta_n & \boxed{\theta_{n,N+1}} \end{vmatrix} \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\theta_{n,N+1}, P_n) \\ \Omega(\Theta_n, \rho_{n,N+1}) & \boxed{\Omega(\theta_{n,N+1}, \rho_{n,N+1})} \end{vmatrix}^{-1} \right. \\ &\quad \left. \begin{vmatrix} \Omega(\Theta_{n-1}, P_{n-1}) & P_n^\dagger \\ \Omega(\Theta_{n-1}, \rho_{n-1,N+1}) & \boxed{\rho_{n,N+1}^\dagger} \end{vmatrix} \right) \begin{vmatrix} \Omega(\Theta_n, P_n) & P_n^\dagger \\ \Theta_n & \boxed{-I} \end{vmatrix} X_n, \end{aligned}$$

Noticing

$$\begin{aligned}
& \left| \begin{array}{cc} \Omega(\Theta_{n-1}, P_{n-1}) & P_n^\dagger \\ \Omega(\Theta_{n-1}, \rho_{n-1, N+1}) & \boxed{\rho_{n, N+1}^\dagger} \end{array} \right| \left| \begin{array}{cc} \Omega(\Theta_n, P_n) & P_n^\dagger \\ \Theta_n & \boxed{-I} \end{array} \right| \\
&= -(\rho_{n, N+1}^\dagger - \Omega(\Theta_{n-1}, \rho_{n-1, N+1})\Omega(\Theta_{n-1}, P_{n-1})^{-1}P_n^\dagger)(I + \Theta_n\Omega(\Theta_n, P_n)^{-1}P_n^\dagger) \\
&= -\rho_{n, N+1}^\dagger + \Omega(\Theta_{n-1}, \rho_{n-1, N+1})\Omega(\Theta_{n-1}, P_{n-1})^{-1}P_n^\dagger \\
&\quad + (\Omega(\Theta_n, \rho_{n, N+1}) - \Omega(\Theta_{n-1}, \rho_{n-1, N+1}))\Omega(\Theta_n, P_n)^{-1}P_n^\dagger \\
&\quad + \Omega(\Theta_{n-1}, \rho_{n-1, N+1})\Omega(\Theta_{n-1}, P_{n-1})^{-1}(\Omega(\Theta_{n-1}, P_{n-1}) - \Omega(\Theta_n, P_n))\Omega(\Theta_n, P_n)^{-1}P_n^\dagger \\
&= -\rho_{n, N+1}^\dagger + \Omega(\Theta_n, \rho_{n, N+1})\Omega(\Theta_n, P_n)^{-1}P_n^\dagger \\
&= - \left| \begin{array}{cc} \Omega(\Theta_n, P_n) & P_n^\dagger \\ \Omega(\Theta_n, \rho_{n, N+1}) & \boxed{\rho_{n, N+1}^\dagger} \end{array} \right|,
\end{aligned}$$

we have

$$X_n[N+2] = - \left| \begin{array}{ccc} \Omega(\Theta_n, P_n) & \Omega(\theta_{n, N+1}, P_n) & P_n^\dagger \\ \Omega(\Theta_n, \rho_{n, N+1}) & \Omega(\theta_{n, N+1}, \rho_{n, N+1}) & \rho_{n, N+1}^\dagger \\ \Theta_n & \theta_{n, N+1} & \boxed{-I} \end{array} \right| X_n.$$

5 Matrix sine-Gordon equation and its kink solutions

It is well known in the commutative case that one may obtain reductions by imposing periodic conditions on the θ_n . Similarly in non-Abelian case, one can make periodic reductions of (1.7). From now on, we only consider the case that X_n is a $d \times d$ matrix and $\alpha_n = I_{d \times d}$ and so (1.7) is

$$(X_{n,x}X_n^{-1})_t + X_{n-1}X_n^{-1} - X_nX_{n+1}^{-1} = 0. \quad (5.1)$$

The simplest such reduction has period 2, that is, we take $X_n = X_{n+2}$ and (5.1) gives the system

$$(X_{0,x}X_0^{-1})_t + X_1X_0^{-1} - X_0X_1^{-1} = 0, \quad (5.2)$$

$$(X_{1,x}X_1^{-1})_t + X_0X_1^{-1} - X_1X_0^{-1} = 0. \quad (5.3)$$

We call this a non-Abelian sinh-Gordon equation since in the commutative case, it will be seen that $X_0 = X_1^{-1} = F_1/F_0$ and then $\theta = 2\log(F_1/F_0)$ satisfies the standard sinh-Gordon equation

$$\theta_{xt} = 4 \sinh \theta.$$

By changing $\theta \rightarrow i\theta$, we can also obtain the sine-Gordon equation

$$\theta_{xt} = 4 \sin \theta.$$

In what follows, we will construct solutions to (5.2)–(5.3) by reduction of the solutions (4.13) of the non-Abelian Toda lattice (5.1). It is clear that (5.1) has vacuum solution $X_n = I$ and (4.13) gives the quasigrammian solutions

$$X_n = - \left| \begin{array}{cc} \Omega(\Theta_n, P_n) & P_n^T \\ \Theta_n & \boxed{-I} \end{array} \right|, \quad (5.4)$$

where $\theta_{n,i}$ and $\rho_{n,i}$ satisfy

$$(\theta_n)_x = \theta_{n-1}, \quad (\theta_n)_t = \theta_{n+1}, \quad (\rho_n)_x = -\rho_{n+1}, \quad (\rho_n)_t = -\rho_{n-1}, \quad (5.5)$$

and Ω is defined by (4.3)–(4.5). We choose the simplest non-trivial solutions of (5.5)

$$\theta_{n,j} = B_j q_j^{-n} e^{q_j x + \frac{1}{q_j} t}, \quad \rho_{n,i} = A_i p_i^n e^{-p_i x - \frac{1}{p_i} t}$$

where A_i and B_j are $d \times d$ matrices and then we obtain

$$\Omega(\theta_{n,j}, \rho_{n,i}) = \delta_{i,j} I + \frac{A_i^t B_j p_i}{q_j - p_i} \left(\frac{p_i}{q_j} \right)^n e^{(q_j - p_i)x + (\frac{1}{q_j} - \frac{1}{p_i})t}.$$

The choice of constant of integration as $\delta_{i,j} I$ is needed to effect the periodic reduction we will shortly make. This can also be written as

$$\Omega(\theta_{n,j}, \rho_{n,i}) = \left(\frac{p_i}{q_j} \right)^n \left(\delta_{i,j} I \left(\frac{q_j}{p_i} \right)^n + \frac{A_i^t B_j p_i}{q_j - p_i} e^{(q_j - p_i)x + (\frac{1}{q_j} - \frac{1}{p_i})t} \right).$$

Now using the invariance of a quasideterminant to scaling of its rows and columns (see e.g. [26]), we get

$$X_n = - \left| \begin{array}{cc} \left(\delta_{i,j} \left(\frac{q_j}{p_i} \right)^n I + \frac{A_i^t B_j p_i}{q_j - p_i} e^{(q_j - p_i)x + (\frac{1}{q_j} - \frac{1}{p_i})t} \right) & (A_i e^{-p_i x - \frac{1}{p_i} t})^T \\ (B_j e^{q_j x + \frac{1}{q_j} t}) & \boxed{-I} \end{array} \right|$$

It is obvious from this expression for X_n that it is 2 periodic when $(\frac{q_1}{p_1})^2 = \dots = (\frac{q_N}{p_N})^2 = 1$, that is, $p_i = -q_i = \lambda_i$ for $i = 1, \dots, N$. Therefore, the non-Abelian sinh-Gordon equation has the solutions

$$X_0 = - \left| \begin{array}{cc} \left(\delta_{i,j} I - \frac{A_i^t B_j \lambda_i}{(\lambda_i + \lambda_j)} e^{-(\lambda_i + \lambda_j)x - (\frac{1}{\lambda_i} + \frac{1}{\lambda_j})t} \right) & (A_i e^{-\lambda_i x - \frac{1}{\lambda_i} t})^T \\ (B_j e^{-\lambda_j x - \frac{1}{\lambda_j} t}) & \boxed{-I} \end{array} \right|, \quad (5.6)$$

$$X_1 = - \left| \begin{array}{cc} \left(-\delta_{i,j} I - \frac{A_i^t B_j \lambda_i}{(\lambda_i + \lambda_j)} e^{-(\lambda_i + \lambda_j)x - (\frac{1}{\lambda_i} + \frac{1}{\lambda_j})t} \right) & (A_i e^{-\lambda_i x - \frac{1}{\lambda_i} t})^T \\ (B_j e^{-\lambda_j x - \frac{1}{\lambda_j} t}) & \boxed{-I} \end{array} \right|. \quad (5.7)$$

From now on we will assume that $A_i = I$ are real and $B_j = i r_j P_j$, where r_j are real scalars, are pure imaginary matrices. In this case, it follows that X_0 and X_1 are complex conjugate to one another. For this reason we introduce

$$X = X_0 = \bar{X}_1 = - \left| \begin{array}{cc} \left(\delta_{i,j} I - \frac{B_j \lambda_i}{(\lambda_i + \lambda_j)} e^{-(\lambda_i + \lambda_j)x - (\frac{1}{\lambda_i} + \frac{1}{\lambda_j})t} \right) & (e^{-\lambda_i x - \frac{1}{\lambda_i} t} I)^T \\ (B_j e^{-\lambda_j x - \frac{1}{\lambda_j} t}) & \boxed{-I} \end{array} \right|. \quad (5.8)$$

Next, we will derive matrix kink solutions for the matrix sine-Gordon equation using the method applied to study the soliton solutions of the matrix KdV equation in [24]. To get a visual representation of the solution we will consider the matrix $W(x, t)$ defined by

$$iW_x = \bar{X}_x \bar{X}^{-1} - X_x X^{-1}. \quad (5.9)$$

We choose this dependent variable so that in the scalar case $W = \theta$, the solution of the sine-Gordon equation.

For $N = 1$, (5.8) gives

$$X = I + B \left(I - \frac{B}{2} e^{-2\lambda x - \frac{2}{\lambda} t} \right)^{-1} e^{-2\lambda x - \frac{2}{\lambda} t}. \quad (5.10)$$

We first assume further that P is a projection matrix (i.e. satisfies $P^2 = P$). This choice allows us to calculate the inverse matrices in the above expression explicitly using the formula

$$(I - aP)^{-1} = I + \frac{aP}{1 - a}, \quad (5.11)$$

where $a \neq 1$ is a scalar and P is any projection matrix.

In this way we find that

$$X = I + \frac{irP}{e^{2\lambda x + \frac{2}{\lambda} t} - ir/2},$$

and hence

$$W_x = \frac{4\lambda P}{\cosh(2\lambda(x + t/\lambda^2 - \phi))}$$

where $\phi = \log(r/2)/2\lambda$. Note also that $X\bar{X} = I$.

Taking one final step, we integrate to obtain the one-kink solution to the matrix sine-Gordon equation

$$W = 4P \arctan(e^{2\lambda(x+t/\lambda^2-\phi)}). \quad (5.12)$$

Remark. For the one-kink solution (5.12), we call the projection matrix P its polarization and ϕ its phase. In the scalar case, if we choose $P = 1$, (5.12) is simply the one-kink solution to the standard sine-Gordon equation.

For $N = 2$, expanding X by the definition (2.1), we can rewrite X as

$$\begin{aligned} X &= I + (B_1 e^{-\lambda_1 x - \frac{1}{\lambda_1} t}, B_2 e^{-\lambda_2 x - \frac{1}{\lambda_2} t}) \left(\delta_{i,j} I - \frac{B_j \lambda_i}{(\lambda_i + \lambda_j)} e^{-(\lambda_i + \lambda_j)x - (\frac{1}{\lambda_i} + \frac{1}{\lambda_j})t} \right)^{-1}_{2 \times 2} \begin{pmatrix} e^{-\lambda_1 x - \frac{1}{\lambda_1} t} I \\ e^{-\lambda_2 x - \frac{1}{\lambda_2} t} I \end{pmatrix} \\ &= I + (L_1 e^{\lambda_1 x + \frac{1}{\lambda_1} t}, L_2 e^{\lambda_2 x + \frac{1}{\lambda_2} t}) \begin{pmatrix} e^{-\lambda_1 x - \frac{1}{\lambda_1} t} I \\ e^{-\lambda_2 x - \frac{1}{\lambda_2} t} I \end{pmatrix} \\ &= I + L_1 + L_2, \end{aligned}$$

and hence

$$\begin{aligned} L_1(e^{2\lambda_1 x + \frac{2}{\lambda_1} t} I - \frac{1}{2} B_1) - \frac{\lambda_2}{\lambda_1 + \lambda_2} L_2 B_1 &= B_1, \\ L_2(e^{2\lambda_2 x + \frac{2}{\lambda_2} t} I - \frac{1}{2} B_2) - \frac{\lambda_1}{\lambda_1 + \lambda_2} L_1 B_2 &= B_2. \end{aligned}$$

In the expressions $B_j = ir_j P_j$, $j = 1, 2$ we assume that P_j are the rank-1 projection matrices

$$P_j = \frac{p_j \otimes q_j}{(p_j, q_j)} = \frac{p_j q_j^T}{p_j^T q_j}$$

and the d -vectors p_j and q_j satisfy the condition $(p_j, q_j) \neq 0$, we can solve for L_1 and L_2 by using (5.11) to obtain

$$\begin{aligned} L_1 &= \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_2 B_2 + (\lambda_1 + \lambda_2) g_2 I) B_1, \\ L_2 &= \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_1 B_1 + (\lambda_1 + \lambda_2) g_1 I) B_2, \end{aligned}$$

where

$$g = (\lambda_1 + \lambda_2)^2 g_1 g_2 + \lambda_1 \lambda_2 r_1 r_2 \alpha, \quad \text{where } \alpha = \frac{(p_1, q_2)(p_2, q_1)}{(p_1, q_1)(p_2, q_2)}$$

and

$$g_j = e^{2\lambda_j \theta_j} - \frac{ir_j}{2} \quad \text{for } j = 1, 2,$$

where

$$\theta_j = x + \frac{1}{\lambda_j^2} t.$$

Therefore

$$X = I + \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_1 B_1 B_2 + \lambda_2 B_2 B_1 + (\lambda_1 + \lambda_2)(g_1 B_2 + g_2 B_1)). \quad (5.13)$$

We now investigate the behaviour of X as $t \rightarrow \pm\infty$. We will use the fact that W is invariant under the transformation $X \rightarrow XC$ for any constant matrix C and assume, without loss of generality, that $0 < \lambda_1 < \lambda_2$. In the calculations that follow, we will demonstrate that kinks emerge from the interaction and undergo phase-shifts as in the scalar case. In addition however, we will see that there are changes of polarization, in other words, amplitudes may also change as a result of the interaction.

First we fix θ_1 . Then $\theta_2 = \theta_1 + (1/\lambda_2^2 - 1/\lambda_1^2)t$ and so as $t \rightarrow -\infty$,

$$X \sim I + \frac{B_1}{g_1} = I + \frac{ir_1 P_1}{e^{2\lambda_1 \theta_1} - \frac{ir_1}{2}}.$$

As $t \rightarrow +\infty$, using the invariance of W , we obtain

$$\begin{aligned} X &\sim I + \frac{2B_2 i}{r_2} - \frac{(\lambda_1 + \lambda_2)(2\lambda_1 B_1 B_2 + 2\lambda_2 B_2 B_1 - (\lambda_1 + \lambda_2)r_2 B_1 i) - 4\alpha\lambda_1\lambda_2 r_1 B_2 i}{r_2(\lambda_1 + \lambda_2)^2 g_1 i - 2\alpha\lambda_1\lambda_2 r_1 r_2} \\ &\sim \left(I - \frac{(\lambda_1 + \lambda_2)(2\lambda_1 B_1 B_2 + 2\lambda_2 B_2 B_1 - (\lambda_1 + \lambda_2)r_2 B_1 i) - 4\alpha\lambda_1\lambda_2 r_1 B_2 i}{r_2(\lambda_1 + \lambda_2)^2 g_1 i - 2\alpha\lambda_1\lambda_2 r_1 r_2} \left(I + \frac{2B_2 i}{r_2} \right) \right) \left(I + \frac{2B_2 i}{r_2} \right) \\ &\sim I + \frac{i\hat{r}_1 \hat{P}_1}{e^{2\lambda_1 \theta_1} - \frac{i\hat{r}_1}{2}}, \end{aligned}$$

where

$$\begin{aligned} \hat{r}_1 &= \frac{r_1(\hat{p}_1, \hat{q}_1)}{(p_1, q_1)}, \quad \hat{P}_1 = \frac{\hat{p}_1 \otimes \hat{q}_1}{(\hat{p}_1, \hat{q}_1)}, \\ \hat{p}_1 &= p_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)} \frac{(p_1, q_2)}{(p_2, q_2)} p_2, \quad \hat{q}_1 = q_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)} \frac{(p_2, q_1)}{(p_2, q_2)} q_2. \end{aligned}$$

This shows that

$$\begin{aligned} W &\sim 4P_1 \arctan(e^{2\lambda_1(\theta_1 - \phi_1^-)}), \quad t \rightarrow -\infty, \\ W &\sim 4\hat{P}_1 \arctan(e^{2\lambda_1(\theta_1 - \phi_1^+)}), \quad t \rightarrow +\infty \end{aligned}$$

where

$$\phi_1^- = \frac{1}{2\lambda_1} \log \frac{r_1}{2}, \quad \phi_1^+ = \frac{1}{2\lambda_1} \log \frac{r_1(\hat{p}_1, \hat{q}_1)}{2(p_1, q_1)}.$$

Similarly, fixing θ_2 , we have

$$\begin{aligned} X &\sim I + \frac{i\hat{r}_2\hat{P}_2}{e^{2\lambda_2\theta_2} - \frac{i\hat{r}_2}{2}}, \quad t \rightarrow -\infty, \\ X &\sim I + \frac{B_2}{g_2} = I + \frac{ir_2P_2}{e^{2\lambda_2\theta_2} - \frac{ir_2}{2}}, \quad t \rightarrow +\infty, \end{aligned}$$

where

$$\begin{aligned} \hat{r}_2 &= \frac{r_2(\hat{p}_2, \hat{q}_2)}{(p_2, q_2)}, \quad \hat{P}_2 = \frac{\hat{p}_2 \otimes \hat{q}_2}{(\hat{p}_2, \hat{q}_2)}, \\ \hat{p}_2 &= p_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)} \frac{(p_2, q_1)}{(p_1, q_1)} p_1, \quad \hat{q}_2 = q_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)} \frac{(p_1, q_2)}{(p_1, q_1)} q_1. \end{aligned}$$

Then

$$\begin{aligned} W &\sim 4\hat{P}_2 \arctan(e^{2\lambda_2(\theta_2 - \phi_2^-)}), \quad t \rightarrow -\infty \\ W &\sim 4P_2 \arctan(e^{2\lambda_2(\theta_2 - \phi_2^+)}), \quad t \rightarrow +\infty, \end{aligned}$$

where

$$\phi_2^- = \frac{1}{2\lambda_2} \log \frac{r_2(\hat{p}_2, \hat{q}_2)}{2(p_2, q_2)}, \quad \phi_2^+ = \frac{1}{2\lambda_2} \log \frac{r_2}{2}.$$

The above calculations show that $W(x, t)$ decomposes into the sum of two kink solutions as $t \rightarrow \pm\infty$ and the j th kink solution propagates with the speed $1/\lambda_j^2$. The phase shifts $\Delta_j = \phi_j^+ - \phi_j^-$ for the kink solutions are

$$\Delta_1 = \frac{1}{2\lambda_1} \log \beta, \quad \Delta_2 = -\frac{1}{2\lambda_2} \log \beta,$$

where

$$\beta = 1 - \frac{4\lambda_1\lambda_2\alpha}{(\lambda_1 + \lambda_2)^2}.$$

Remark. In a similar way to the matrix KdV equation in [24], we find that the matrix amplitude of the first kink solution changes from $4P_1$ to $4\hat{P}_1$ and the matrix amplitude of the other one changes from $4\hat{P}_2$ to $4P_2$ as t changes from $-\infty$ to $+\infty$. If $(p_1, q_2) = 0$ ($P_2P_1 = 0$) or $(p_2, q_1) = 0$ ($P_1P_2 = 0$), there is no phase-shift, however the amplitudes may change. In the case that both $P_1P_2 = P_2P_1 = 0$, there is neither phase-shift nor change in amplitude and so the kink solutions have trivial interaction.

To illustrate the above, we will consider the case $d = 2$, i.e. the 2×2 matrix sine-Gordon equation. We choose $\lambda_1 = 1$, $\lambda_2 = 2$, $r_1 = r_2 = 1$, and

$$P_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{3} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$

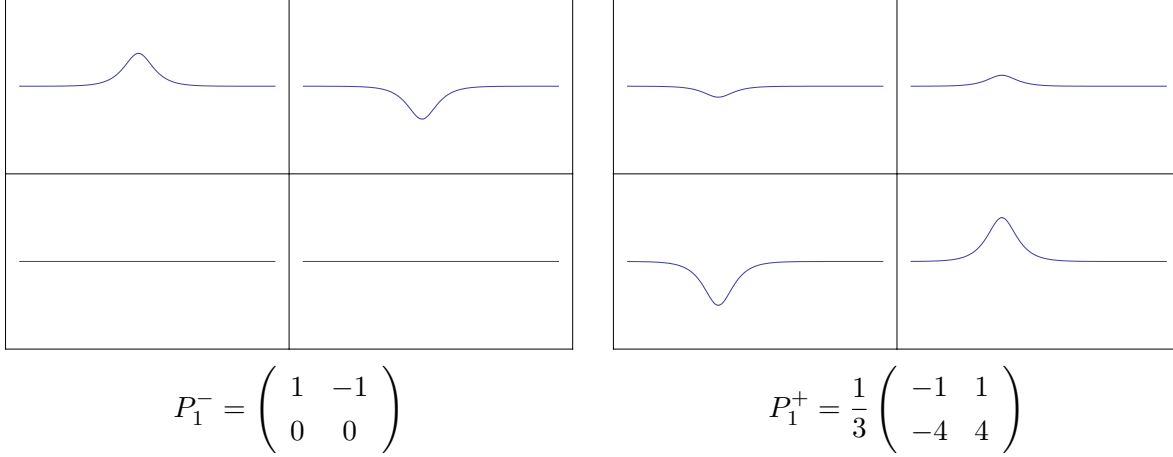


Figure 1: Asymptotic forms for kink 1

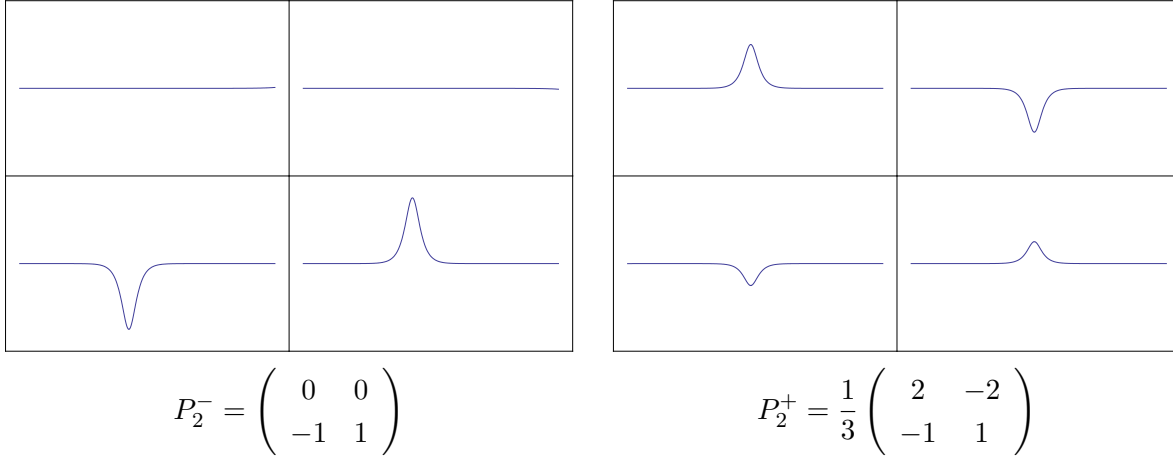


Figure 2: Asymptotic forms for kink 2

The analysis above shows that $P_1^- = P_1$, $P_2^+ = P_2$ and

$$P_1^+ = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}, \quad P_2^- = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

For convenience, rather than plotting the kink W given by (5.9), we plot the derivative W_x and refer to it as a soliton. In Figure 1, the asymptotic forms of the matrix soliton 1 are plotted as $t \rightarrow \pm\infty$. The first plot exhibits the amplitudes given by P_1^- and the second, those of P_1^+ . Similarly, in Figure 2, we show the same plots for soliton 2.

6 Conclusions

In this paper, we have considered a generalized non-Abelian Toda lattice and presented quasiwronskian and quasigrammian solutions obtained means of by Darboux transformations and binary Darboux transformations respectively. Then we imposed a 2-periodic reduction on the non-Abelian Toda lattice to derive a noncommutative sine-Gordon equation. By using a method similar to that developed in [24] for the matrix KdV equation, we obtained kink solutions for the matrix sine-Gordon equation from the quasigrammian solutions of the non-Abelian Toda lattice. Then we

investigated the interaction properties of matrix kink solutions. It is known [27] that the change of matrix amplitude of solitons for the matrix KdV equation gives rise to a Yang-Baxter map. It would be interesting to investigate whether there is a similar result for the matrix sine-Gordon equation.

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